Effective wave numbers for thermo-viscoelastic media containing random configurations of spherical scatterers

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The dispersion relation is derived for the coherent waves in fluid or elastic media supporting viscous and thermal effects and containing randomly distributed spherical scatterers. The formula obtained is the generalization of Lloyd and Berry's [Proc. Phys. Soc. Lond. **91**, 678-688, 1067], the latter being limited to fluid host media, and it is the three-dimensional counterpart of that derived by Conoir and Norris [Wave Motion **47**, 183-197, 2010] for cylindrical scatterers in an elastic host medium.

I. INTRODUCTION

This study considers wave propagation through a homogeneous isotropic host medium containing a large number of randomly and uniformly located spherical scatterers. It is well known that for small enough concentrations the physical medium may be replaced by an effective homogeneous medium in which coherent waves propagate. These waves correspond to the average, over all possible locations of the scatterers, of the multiply-scattered field in the actual host medium. Our concern here is to find the effective wavenumbers of the coherent waves at low frequency and low concentration of spheres. This was achieved some time ago by Lloyd and Berry[1] for the case of acoustic waves with $P = 1$ (P: number of waves that propagate in the host medium). A new and clear derivation of Lloyd and Berry's formula was recently given by Linton and Martin[2] using a procedure developed earlier for cylindrical scatterers in Ref.[3]. They considered a uniform concentration of scatterers satisfying the hole correction of Fikioris and Waterman[4], and found solutions of the implicit dispersion relation as an expansion in terms of the concentration of scatterers, up to order 2, under the low frequency assumption. Generalization of Linton and Martin's formula for cylinders in a fluid[3] has been given by Conoir and Norris[5] for cylinders in an elastic solid with $P = 2$ (compressional and shear waves propagate in elastic solids). The present paper is an extension of the results of Linton and Martin[2] to homogeneous isotropic host media supporting viscous and thermal (damped) waves for which $P = 3$. Examples of such media include viscous fluids[6,7] with compressional and transverse waves ($P = 2$), and thermoelastic solids[8–10] in which compressional, transverse and thermal waves propagate ($P = 3$). The results obtained here can serve as a starting point to generalize effective medium theories that include multiple wave interactions, a broad frequency range, and finite concentration levels, such as the ECAH model (Epstein, Carhart, Allegra and Hawley)[11].

The analysis begins in section II with a description of the scalar potentials that are used to describe the wave propagation in the host medium. The choice of the potentials depends on conditions of symmetries which are discussed. Multiple scattering theory is used in section III to derive the modal equations (Lorentz-Lorenz laws) relating the amplitudes of the P-wave types. In section IV the modal equations are written in matrix form which allows us to obtain a compact form of the dispersion equation governing the coherent waves. The low concentration and low frequency assumptions are then introduced in section V and expansions of the effective wavenumbers up to second order in concentration are derived. The effective wavenumbers are expressed in terms of series, and, for the faster wave, acoustic wave in viscous fluids and compressional wave in viscoelastic solids, the wave number is given by an integral relation that generalizes the one of Lloyd and Berry[1].

II. THE DIFFERENT TYPES OF COHERENT WAVES

We first define the multiple scattering problem for identical spheres in homogeneous isotropic host media in which three type of waves are present, compressional (c), viscous or shear (s) and thermal (th). The multiple scattering theory used here follows the lines of that first developed by Fikioris & Waterman[4] for acoustic waves.

A. The different types of waves in the host medium

The dynamic displacement field $\vec{u}$ in the host medium may always be decomposed as $\vec{u} = \vec{\nabla}\psi_L + \vec{\nabla} \times \vec{\psi}_R$, with ψ_L the scalar potential for longitudinal (compressional or thermal) wave motion, and $\vec{\psi}_R$ the vector potential associated with rotational waves[9]. Each wave type propagates with its own complex wavenumber k_p, with p an integer numbering the wave ($1 \leq p \leq P$). The rotational wave field itself can generally be partitioned into a shear and a transverse wave, corresponding to the decomposition of the vector potential $\vec{\psi}_R$ as the sum of two orthogonal vectors $\vec{\psi}_R = \vec{\nabla} \times (\psi r \widehat{e_r}) + (1/k_p)\,\vec{\nabla} \times \vec{\nabla} \times (\chi r \widehat{e_r})$ with $\widehat{e_r}$ the unit vector along the radial coordinates[12]. The Debye potentials ψ and χ are associated with the shear (s) and transverse (T) waves, respectively. As a consequence, as many as four distinct wave types can propagate in the host medium ($p = c, th, s, T$).

Our objective here is to determine the dispersion relations of the coherent waves in the presence of spheres that are randomly and uniformly located in the host medium. *A priori*, there should be one coherent wave for each longitudinal wave, and one for each shear and each transverse wave. This is a natural hypothesis if the scatterers are scarce, that is of low volumetric concentration. It is important to note that the dispersion equations are a characteristic of the effective medium itself, and, as such, they should not depend on the type of the incident plane wave, which will be supposed from now on to be a compressional wave. This choice implies symmetries on the scattering by one scatterer[9,10], which, in turn, have implications on the form of the analytical expressions for the potentials associated with the coherent waves propagating in the effective medium.

B. Symmetries

The scatterers are assumed to be randomly and uniformly located in the semi-infinite region $z > 0$ of the host medium, with an harmonic compressional plane wave incident on the boundary $z = 0$. Coherent wavenumbers do not depend on the angle of incidence of the incident wave[2], and so with no loss in generality we assume normal incidence. Let us first consider scattering by a single sphere of an incident compressional plane wave propagating in the z direction. Due to the symmetry of the sphere, the scattered fields do not depend on the azimuthal angle φ, with spherical coordinates defined by the sphere center $((x, y, z) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta))$[9,10]. Consequently, the scattered transverse T wave with pure azimuthal displacement is zero. The question now is whether a transverse coherent wave can arise from the multiple scattering process. The answer is not straightforward, because the independence of a field with respect to the azimuthal angle in a given spherical coordinate system does not guarantee independence of that same field with respect to the azimuthal angle in another spherical coordinate system. This means that each multiply scattered field is a combination of longitudinal, shear and transverse waves. However, if the scatterers are uniformly distributed, the average introduces an homogenization of the different fields in the medium, and the effective fields exciting a given sphere should not depend on the azimuthal angle φ, and, thus, the coherent transverse T field should vanish.

Another way of considering the symmetry is to bear in mind that coherent waves result mainly from constructive interferences of waves traveling from one sphere to another along a "straight line" (the direction of propagation of the coherent waves). The effective coherent fields maintain the same symmetry as that for the field scattered by a single sphere. We therefore conclude that the coherent fields do not depend on the azimuthal direction φ in the (x, y) plane perpendicular to the direction of the incident plane wave. Mathematically, this result is exact up to order 2 in concentration, as shown in Appendix A. In the following, the longitudinal fields are described by scalar potentials of displacement, and the shear fields by the only non-zero component (azimuthal component) of the vectorial potential of displacement related to shear waves. These scalar quantities are denoted by $\varphi^{(p)}$, with p denoting again the type of the wave ($p = c, s, th$).

III. MULTIPLE SCATTERING EQUATIONS

Harmonic wave motion is considered with time dependence $\exp(-i\omega t)$ understood. The notation $c = 1$, $s = 2$ and $th = 3$ is employed, similar to that used in Ref.[5]. The potential function $\varphi_S^{(p)}(\vec{r}; \vec{r}_j)$ represents the wave of type p scattered by a target centered at $\vec{r}_j$ and observed at $\vec{r}$; the potential $\varphi_E^{(p)}(\vec{r}; \vec{r}_j)$ denotes the field of type p that

excites a scatterer centered at $\vec{r_j}$ and observed at $\vec{r}$. The following fundamental identity, which is a straightforward generalization of Eqs. (6) from Ref.[5], provides the integral equation governing the coherent fields, denoted by brackets, in the presence of a uniform and random array of identical scatterers:

$$\langle \varphi_E^{(p)}(\vec{r};\vec{r}_1)\rangle = \delta_{p1}\varphi_{inc}^{(1)}(\vec{r}) + \sum_{q=1}^{P}\int \mathrm{d}\,\vec{r}_j\, n(\vec{r_j},\vec{r}_1) T^{qp}(\vec{r}_j)\langle \varphi_E^{(q)}(\vec{r};\vec{r}_j)\rangle. \tag{1}$$

The integration in Eq. (1) is over the semi-infinite region ($z > 0$) containing the spherical scatterers. The function $n(\vec{r_j},\vec{r}_1)$ is the conditional number density of spheres at $\vec{r_j}$ if one is known to be at $\vec{r}_1$, see Ref.[5]. In the following we assume a constant density n_0 of scatterers of radius a, and conditional number density given by the *hole correction*

$$n(\vec{r},\vec{r}_j) = \begin{cases} n_0 & \text{for} \quad |\vec{r}-\vec{r}_j| > b, \quad \text{with} \quad b > 2a. \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The incident compressional wave is assumed to be a damped plane wave

$$\varphi_{inc}^{(1)}(\vec{r}) = e^{ik_p z} \ \text{ with } \ k_p = k_p^{'} + ik_p^{''} \ \ (k_p^{''} > 0). \tag{3}$$

The potential functions are expressed as infinite series of spherical harmonics that respect the underlying symmetry. For the reasons described before they do not depend on the azimuthal direction φ, and so

$$\langle \varphi_E^{(p)}(\vec{r},\vec{r}_j)\rangle = \sum_{n=0}^{+\infty} A_n^{(p)}(\vec{r}_j)\, j_n(k_p\rho_j)P_n\big(\cos\theta(\vec{\rho}_j)\big) \quad \text{with} \quad \vec{\rho}_j \equiv \vec{r}-\vec{r}_j. \tag{4}$$

It proves useful to introduce shorthand notation for spherical harmonics (not to be confused with ordinary Bessel and Hankel function notation)

$$J_n(k_p,\vec{\rho}_j) = j_n(k_p\rho_j)P_n\big(\cos\theta(\vec{\rho}_j)\big), \quad H_n(k_p,\vec{\rho}_j) = h_n^{(1)}(k_p\rho_j)P_n\big(\cos\theta(\vec{\rho}_j)\big), \tag{5}$$

plus the following definition for the action of the transition operators T^{qp} on a spherical harmonic[9,10]

$$T^{qp}(\vec{r}_j)\, J_n(k_p,\vec{\rho}_j) = T_n^{qp}\, H_n(k_p,\vec{\rho}_j). \tag{6}$$

The modal coefficients T_n^{qp} in Eq. (6) can be numerically calculated following the procedures of Refs.[9,10]. Equation (1) can now be expressed, using Eqs. (3) and (4),

$$\sum_{n=0}^{+\infty} A_n^{(p)}(\vec{r}_1)J_n(k_p,\vec{\rho}_1) = \delta_{p1}e^{ik_pz_1} + \sum_{n=0}^{+\infty}\sum_{q=1}^{P}\int \mathrm{d}\,\vec{r}_j\, n(\vec{r_j},\vec{r}_1)A_n^{(q)}(\vec{r}_j)T_n^{qp}H_n(k_p,\vec{\rho}_j). \tag{7}$$

Decomposition of the damped incident plane wave into spherical harmonics in the coordinate system centered at $\vec{r}_1$ leads to

$$\begin{aligned}\sum_{n=0}^{+\infty}\big[A_n^{(p)}(\vec{r}_1) - \delta_{p1}i^n(2n+1)e^{ik_pz_1}\big]J_n(k_p,\vec{\rho}_1)& \\ = \sum_{n=0}^{+\infty}\sum_{q=1}^{P}\int \mathrm{d}\,\vec{r}_j\, n(\vec{r_j},\vec{r}_1)A_n^{(q)}(\vec{r}_j)T_n^{qp}H_n(k_p,\vec{\rho}_j).&\end{aligned} \tag{8}$$

The addition theorem[13] allows us to write the series in the right hand side of Eq. (8) as a function of the coordinates centered on $\vec{r}_1$,

$$\begin{aligned}H_n(k_p,\vec{\rho}_j) = &\sum_{\nu=0}^{+\infty}\sum_{\mu=-\nu}^{+\nu}\sum_{\ell=0}^{+\infty}(-1)^{\mu}\, i^{\nu+\ell-n}(2\nu+1)G(0,n;-\mu,\nu;\ell)e^{i\mu\varphi(\vec{\rho}_1)}e^{-i\mu\varphi(\vec{r_{1j}})} \\ &\times h_n^{(1)}(k_pr_{1j})P_\ell^{-\mu}\big(\cos\theta(\vec{r_{1j}})\big)j_\nu(k_p\rho_1)P_n^{\mu}\big(\cos\theta(\vec{\rho}_1)\big) \quad \text{with} \quad \vec{r_{1j}} = \vec{r_1}-\vec{r_j},\end{aligned} \tag{9}$$

where the Gaunt coefficients $G(0,n;-\mu,\nu;m)$ are defined by

$$P_n^m\big(\cos\theta\big)P_\nu^\mu\big(\cos\theta\big)=\sum_{\ell=0}^{\infty}G(m,n;\mu,\nu;\ell)P_\ell^{m+\mu}\big(\cos\theta\big). \tag{10}$$

By assumption, the incident damped plane wave impinges on the $(z=0)$ interface at normal incidence, and gives rise to damped coherent waves that propagate and are attenuated in the same direction z. Accordingly, we seek coherent plane wave solutions obeying the Snell-Descartes laws of refraction. We thus search for the solutions of Eq. (8) in the form

$$A_n^{(q)}(\vec{r_j})=i^n(2n+1)\sum_{p=1}^{P}A_n^{(qp)}e^{i\xi_p z_j}. \tag{11}$$

Substituting from Eqs. (9) and (11) into Eq. (8), making the change of variables of integration $\vec{r_j}=\vec{r_1}-\vec{r_{1j}}$, and noting that the integration over $d\varphi_{1j}=d\varphi(\vec{r_{1j}})$ with $d\vec{r_{1j}}=r_{1j}^2\sin\theta_{1j}dr_{1j}d\theta_{1j}d\varphi_{1j}$ gives rise to zero except for $\mu=0$ because of the term $\exp\big[-i\mu\varphi(\vec{r_{1j}})\big]$, yields

$$A_n^{(pk)}e^{i\xi_k z_1}=\delta_{p1}e^{ik_p z_1}+\sum_{q=1}^{P}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}i^\ell(2\nu+1)T_\nu^{qp}A_\nu^{(qk)}G(0,\nu;0,n;\ell)I_\ell^{(p)}(\xi_k), \tag{12}$$

where the quantity $I_\ell^{(p)}$ is

$$I_\ell^{(p)}(\xi_k)=\int \mathrm{d}\,\vec{r}_j\,n(\vec{r_j},\vec{r}_1)e^{i\xi_k z_j}h_\ell^{(1)}(k_p r_{1j})P_\ell\big(\cos\theta(\vec{r_{1j}})\big). \tag{13}$$

Taking into account the hole correction (2), and following the analysis of Ref.[4], we obtain

$$I_\ell^{(p)}(\xi_k)=\frac{2n_0\pi i^\ell}{\xi_k-k_p}\left[\frac{2b}{\xi_k+k_p}N_\ell^{(p)}(\xi_k)e^{i\xi_k z_1}+\frac{i}{k_p^2}e^{ik_p z_1}\right]\quad\text{with} \tag{14a}$$

$$N_\ell^{(p)}(\xi_k)=\xi_k b j_\ell^{'}(\xi_k b)h_\ell^{(1)}(k_p b)-k_p b j_\ell(\xi_k b)h_\ell^{(1)'}(k_p b). \tag{14b}$$

Inserting Eq. (14) into Eq. (12), and equating the coefficients of $\exp(i\xi_k z_1)$ to zero gives what is known as the Lorentz-Lorenz law (equating the coefficients of $\exp(ik_p z_1)$ to zero gives the extinction theorem). Finally, we obtain the equations for the amplitudes $A_n^{(pk)}$,

$$A_n^{(pk)}=\frac{4n_0\pi b}{\xi_k^2-k_p^2}\sum_{q=1}^{P}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}(-1)^\ell(2\nu+1)T_\nu^{qp}A_\nu^{(qk)}N_\ell^{(p)}(\xi_k)G(0,\nu;0,n;\ell) \tag{15}$$

with $1\leqslant p\leqslant P$ and $1\leqslant k\leqslant P$.

IV. MATRIX FORM OF THE MODAL EQUATION

We now seek to write Eq. (15) in matrix form in the same way as in Ref.[5,14]. With no loss of generality, consider a given effective wave number ξ_k $(k=c,s,th)$. Equation (15) can therefore be written in simplified form by dropping the index k $(\xi_k=\xi)$. Define the non-dimensional parameters

$$y_p=\xi^2-k_p^2\quad\text{and}\quad\varepsilon=-4in_0\,, \tag{16}$$

and the use of the relation[15]

$$(-1)^\ell G(0,\nu;0,n;\ell)=(-1)^{\nu+n}G(0,\nu;0,n;\ell) \tag{17}$$

implies $(1\leqslant p\leqslant P\ )$

$$A_n^{(p)}-\frac{i\varepsilon\pi b}{y_p}\sum_{q=1}^{P}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}(-1)^{\nu+n}(2\nu+1)T_\nu^{qp}A_\nu^{(q)}N_\ell^{(p)}(\xi)G(0,\nu;0,n;\ell)=0. \tag{18}$$

In order to reach our objective, we define the infinitely long vectors $|e\rangle$, $\langle e| = |e\rangle^t$ and square matrices $\mathbf{T}^{qp}$, $\bar{\mathbf{Q}}^{(p)}$ from their components, $e_n = (-1)^n$ and

$$T^{qp}_{n\nu} = \delta_{n\nu}(2\nu+1)T^{qp}_{\nu}, \quad \bar{Q}^{(p)}_{n\nu}(\xi) = \frac{\pi}{k_p y_p}\big[\, ik_p b \sum_{\ell=0}^{+\infty} N^{(p)}_{\ell}(\xi) G(0,\nu;0,n;\ell) - 1\,\big](|e\rangle\langle e|)_{n\nu} \ . \tag{19}$$

We also introduce the unknown vectors $|a_p\rangle$ with components $A^{(p)}_n$, the vectors $|e_p\rangle$ consisting in the combination of P zero vectors except at the p$^{\text{th}}$ place where their components are equal to $\sqrt{\pi/k_p}\,|e\rangle$, the $|a\rangle$ vector that is a collection of the $|a_p\rangle$ vectors, and the block matrices

$$\mathbf{I}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} \mathbf{T}^{11} & \mathbf{T}^{21} & \mathbf{T}^{31} \\ \mathbf{T}^{12} & \mathbf{T}^{22} & \mathbf{T}^{32} \\ \mathbf{T}^{13} & \mathbf{T}^{23} & \mathbf{T}^{33} \end{bmatrix}, \qquad \bar{\mathbf{Q}} = \begin{bmatrix} \bar{\mathbf{Q}}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{Q}}^{(3)} \end{bmatrix}. \tag{20}$$

Using these definitions the system of equations (18) can be replaced by the equivalent condition

$$\big\{\mathbf{I}_P - \varepsilon\bar{\mathbf{Q}}\mathbf{T} - \varepsilon\sum_{p=1}^{P}\frac{|e_p\rangle\langle e_p|}{y_p}\mathbf{T}\big\}\,|a\rangle = |0\rangle. \tag{21}$$

The matrices and vectors as defined are based on the assumption $P = 3$. For smaller values of P, or equivalently, no coupling between the wave types ($\mathbf{T}^{qp} = \mathbf{T}^{qq}\delta_{qp}$), Eq. (21) decouples into separate equations for each wave type, each analogous to the acoustic ($P = 1$) case.

Equation (21) embodies the multiple scattering of the P wave types in a single consistency relation. Its structure has been specifically chosen so that it is the same for spherical and cylindrical geometries[5] whatever the number of waves ($P = 1, 2, 3$). We can therefore follow the procedures as developed in Ref.[5,14] for the cylindrical case. After multiplication on the left hand side by $\mathbf{T}^{\frac{1}{2}}$ and introduction of the following quantities

$$|f_p\rangle = \mathbf{T}^{\frac{1}{2}}|e_p\rangle, \quad \langle f_p| = \langle e_p|\mathbf{T}^{\frac{1}{2}}, \quad |b\rangle = \mathbf{T}^{\frac{1}{2}}|a\rangle \ \text{ and } \ \mathbf{Q} = \mathbf{T}^{\frac{1}{2}}\bar{\mathbf{Q}}\mathbf{T}^{\frac{1}{2}}, \tag{22}$$

we get

$$\big\{\mathbf{I}_P - \varepsilon(\mathbf{I}_P - \epsilon\mathbf{Q})^{-1}\sum_{p=1}^{P}\frac{|f_p\rangle\langle f_p|}{y_p}\big\}\,|b\rangle = |0\rangle. \tag{23}$$

The modal equation is obtained by setting to zero the determinant associated with the infinite homogeneous system of equations Eq. (21)[4,16]. This can be written in a much simpler form by writing Eq. (23) in the form

$$\big\{\mathbf{I}_P + \sum_{p=1}^{P}|g_p\rangle\langle f_p|\big\}\,|b\rangle = |0\rangle, \quad \text{where} \ \ |g_p\rangle \equiv -\varepsilon y_p^{-1}(\mathbf{I}_P - \epsilon\mathbf{Q})^{-1}|f_p\rangle. \tag{24}$$

Noting that the matrix in (24) is the sum of the infinite dimensional identity plus a matrix of rank P, the determinant can therefore be reduced to one for a P-dimensional matrix through the use of the identity

$$\det\Big(\mathbf{I}_P + \sum_{p=1}^{P}|g_p\rangle\langle f_p|\Big) = \det\big(\mathbf{i}_P + \mathbf{m}\big) \quad \text{where} \ \ m_{qp} = \langle g_q|f_p\rangle, \tag{25}$$

and $\mathbf{i}_P$ is the identity matrix of dimension P. Setting the determinant in Eq. (25) to zero yields

$$\begin{vmatrix} 1-\epsilon\frac{M_{11}}{y_1} & -\epsilon\frac{M_{21}}{y_2} & -\epsilon\frac{M_{31}}{y_3} \\ -\epsilon\frac{M_{12}}{y_1} & 1-\epsilon\frac{M_{22}}{y_2} & -\epsilon\frac{M_{32}}{y_3} \\ -\epsilon\frac{M_{13}}{y_1} & -\epsilon\frac{M_{23}}{y_2} & 1-\epsilon\frac{M_{33}}{y_3} \end{vmatrix} = 0, \tag{26}$$

or equivalently

$$\begin{vmatrix} y_1 - \epsilon M_{11} & -\epsilon M_{21} & -\epsilon M_{31} \\ -\epsilon M_{12} & y_2 - \epsilon M_{22} & -\epsilon M_{32} \\ -\epsilon M_{13} & -\epsilon M_{23} & y_3 - \epsilon M_{33} \end{vmatrix} = 0. \tag{27}$$

where the matrix elements are

$$\begin{aligned} M_{qp}(\xi) &= \langle f_q | (\mathbf{I}_P - \epsilon \mathbf{Q})^{-1} | f_p \rangle \\ &= \langle e_q | \mathbf{T} | e_p \rangle + \sum_{n=1}^{+\infty} \langle e_q | \big(\mathbf{T} \bar{\mathbf{Q}} \mathbf{T} \big)^n | e_p \rangle \varepsilon^n \\ &= M_{qp}^{(0)} + \sum_{n=1}^{+\infty} M_{qp}^{(n)} \varepsilon^n. \end{aligned} \tag{28}$$

This indicates that the elements of the modal equation Eq. (27) can be calculated without evaluation of the square root matrix $\mathbf{T}^{\frac{1}{2}}$. It is also evident that the above reduction in the size of the system determinant is valid as long as the matrix $\mathbf{I}_P - \epsilon \mathbf{Q}$ is non-singular, which is certainly valid for small ε. Equation (27) is the fundamental equation for determining the coherent wavenumbers ξ_p.

V. ASYMPTOTIC SOLUTIONS OF THE WAVENUMBER EQUATION

This section considers asymptotic expansions of the solutions, valid in different limits: first for low concentration and then for low frequency.

A. Low concentration expansion

Rather than working with the wavenumber directly it is more convenient to expand the solutions of Eq. (27) about one of the three leading order solutions $y_p = 0$ ($\xi = k_p$). The non-dimensional parameters ϵ is small at low concentration, $|\epsilon| \ll 1$, and we therefore assume a formal asymptotic expansion in ϵ:

$$y_p = \varepsilon y_p^{(1)} + \varepsilon^2 y_p^{(2)} + \dots \,. \tag{29}$$

Inserting the asymptotic expansion into the modal equation (27) provides (*cf.* Appendix B)

$$y_p^{(1)} = M_{pp}^{(0)}(k_p), \tag{30a}$$

$$y_p^{(2)} = M_{pp}^{(1)}(k_p) + \sum_{q \neq p} \frac{M_{pq}^{(0)}(k_p) M_{qp}^{(0)}(k_p)}{k_p^2 - k_q^2}. \tag{30b}$$

It follows from Eqs. (19) and (28) that

$$M_{qp}^{(0)} = \frac{\pi}{\sqrt{k_q k_p}} \sum_{n=0}^{\infty} (2n+1) T_n^{qp}, \tag{31a}$$

$$M_{pp}^{(1)}(k_p) = \frac{\pi}{k_p} \sum_{q=1}^{3} \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} (-1)^{n+\nu} (2n+1)(2\nu+1) T_n^{qp} \bar{Q}_{n\nu}^{(q)}(k_p) T_\nu^{pq}, \tag{31b}$$

with

$$
\begin{aligned}
\bar{Q}^{(q)}_{n\nu}(k_p) =& \frac{i\pi b}{k_p^2-k_q^2}(-1)^{n+\nu}\Big\{\frac{i}{k_q b}+\sum_{\ell=0}^{+\infty}G(0,\nu;0,n;\ell) \\
&\times\big[k_p b j_\ell'(k_p b)h_\ell^{(1)}(k_q b)-k_q b j_\ell(k_p b)h_\ell^{(1)'}(k_q b)\big]\Big\},\quad q\neq p,
\end{aligned} \tag{32a}
$$

$$
\begin{aligned}
\bar{Q}^{(p)}_{n\nu}(k_p) =& -\frac{i\pi b^2}{2k_p}(-1)^{n+\nu}\sum_{\ell=0}^{+\infty}G(0,\nu;0,n;\ell)\Big\{j_\ell'(k_p b)\big(h_\ell^{(1)}(k_p b)+k_p b h_\ell^{(1)'}(k_q b)\big) \\
&+\frac{1}{k_p b}\big[(k_p b)^2-\ell(\ell+1)\big]j_\ell(k_p b)h_\ell^{(1)}(k_p b)\Big\}
\end{aligned} \tag{32b}
$$

where the following relation has been taken into account,

$$
\sum_{\ell=0}^{+\infty}G(0,\nu;0,n;\ell)=1. \tag{33}
$$

B. Low frequency expansion

The long wavelength limit is defined as $k_p b\to 0$, in which case Eqs. (32) reduce to (see Eq. (33))

$$
\bar{Q}^{(q)}_{n\nu}(k_p) = \frac{\pi}{k_q(k_p^2-k_q^2)}(-1)^{n+\nu}\sum_{\ell=0}^{+\infty}G(0,\nu;0,n;\ell)\,\big[\big(\frac{k_p}{k_q}\big)^{\ell}-1\big],\quad q\neq p, \tag{34a}
$$

$$
\bar{Q}^{(p)}_{n\nu}(k_p) = \frac{\pi}{2k_p^3}(-1)^{n+\nu}\sum_{\ell=0}^{+\infty}\ell\,G(0,\nu;0,n;\ell). \tag{34b}
$$

The low frequency version of Eq. (31b) is

$$
\begin{aligned}
M^{(1)}_{pp}(k_p) =& \frac{\pi^2}{2k_p^4}\sum_{n=0}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}(2n+1)(2\nu+1)\,G(0,\nu;0,n;\ell) \\
&\times\Big\{\ell\,T_n^{pp}T_\nu^{pp}+\sum_{q\neq p}\frac{2k_p^3}{k_q(k_p^2-k_q^2)}\big[\big(\frac{k_p}{k_q}\big)^{\ell}-1\big]T_n^{qp}T_\nu^{pq}\Big\}.
\end{aligned} \tag{35}
$$

and hence the low frequency expansion of the effective wavenumbers are

$$
\begin{aligned}
\frac{\xi_p^2}{k_p^2} =& 1-4i\pi\frac{n_0}{k_p^3}\sum_{n=0}^{\infty}(2n+1)T_n^{pp}-\frac{8\pi^2 n_0^2}{k_p^6}\sum_{n=0}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}(2n+1)(2\nu+1)\,G(0,\nu;0,n;\ell) \\
&\times\Big\{\ell\,T_n^{pp}T_\nu^{pp}+\sum_{q\neq p}\frac{2k_p^3}{k_q(k_p^2-k_q^2)}\big(\frac{k_p}{k_q}\big)^{\ell}\,T_n^{qp}T_\nu^{pq}\Big\}.
\end{aligned} \tag{36}
$$

If the host medium is an ideal fluid, i.e. $P=1$ as in Ref.[2], Eq. (36) simplifies to

$$
\begin{aligned}
\frac{\xi_p^2}{k_p^2} =& 1-4i\pi\frac{n_0}{k_p^3}\sum_{n=0}^{\infty}(2n+1)T_n^{pp} \\
&-\frac{8\pi^2 n_0^2}{k_p^6}\sum_{n=0}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty}(2n+1)(2\nu+1)G(0,\nu;0,n;\ell)\,\ell T_n^{pp}T_\nu^{pp},
\end{aligned} \tag{37}
$$

which is exactly the same relation as that obtained by combining Eqs. (1.2,4.29, 4.33,C.4) from Ref.[2]. Equation (36) contains additional terms which are not in Eq. (37) and are clearly connected to the coupling between the compressional, shear and thermal waves ($q\neq p$). These coupling terms are neglected in the ECAH model (Epstein, Carhart, Allegra and Hawley)[11]. Equation (36) can therefore serve as the starting point for further developments, such as the calculation of the Rayleigh limit, for example, and is the principal result of the paper.

C. Generalization of the Lloyd and Berry formula

The advantage of the Lloyd and Berry formula is that it expresses the wavenumber at low concentration in terms of the far-field scattering function only, rather than the T-matrix elements. In the present context, this requires that we express series in (36) as integrals of the far-field scattering functions defined as follows

$$f^{qp}(\theta) = \sum_{n=0}^{+\infty} (2n+1) T_n^{qp} P_n(\cos\theta). \tag{38}$$

The analogous acoustic problem involves only the terms in Eq. (37) which have been shown to be equivalent to the following expansion in the concentration[2]

$$\xi_p^2 = k_p^2 + \delta_1 n_0 + \delta_2 n_0^2, \tag{39}$$

with

$$\delta_1 = -\frac{4i\pi}{k_p} f^{pp}(0), \tag{40a}$$

$$\delta_2 = \frac{4\pi^2}{k_p^4}\Big\{ \big[f^{pp}(0)\big]^2 - \big[f^{pp}(\pi)\big]^2 + \int_0^\pi \frac{\mathrm{d}\,\theta}{\sin(\frac{\theta}{2})} \frac{\mathrm{d}}{\mathrm{d}\,\theta}[f^{pp}(\theta)]^2 \Big\}, \tag{40b}$$

where δ_2 in Eq. (40b) is the formula initially given by Lloyd and Berry[1]. In order to express the general coupled wave problem in a similar form it is necessary to represent the final series in Eq. (36) ($q \neq p$) as integrals of the far-field scattering functions.

The relevant series that appears in the multi-wave system is (see Eq. (36))

$$S(\kappa) = \sum_{n=0}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty} (2n+1)(2\nu+1) T_n^{qp} T_\nu^{pq} \kappa^\ell G(0,\nu;0,n;\ell) \quad \text{where} \quad \kappa = \frac{k_p}{k_q}. \tag{41}$$

Two cases need to be considered depending upon whether $|\kappa| < 1$ or $|\kappa| > 1$. For the case in which $|\kappa| < 1$, we can define the function

$$g(\kappa,\theta) = \sum_{m=0}^{+\infty} \kappa^m (2m+1) P_m(\cos\theta). \tag{42}$$

The use of the following two relations (*cf.* Ref.[15])

$$\int_0^\pi P_n(\cos\theta) P_m(\cos\theta) \sin\theta \,\mathrm{d}\,\theta = \frac{2}{2n+1}\,\delta_{nm}, \tag{43a}$$

$$\sum_{n=0}^{+\infty} \kappa^n P_n(\cos\theta) = \frac{1}{(1-2\kappa\cos\theta+\kappa^2)^{\frac{1}{2}}} \quad \text{with} \quad |\kappa| < 1, \tag{43b}$$

then provides, respectively,

$$\kappa^n = \frac{1}{2}\int_0^\pi g(\kappa,\theta) P_n(\cos\theta) \sin\theta \,\mathrm{d}\,\theta, \tag{44a}$$

$$g(\kappa,\theta) = \big[1 + 2\kappa \frac{d}{d\kappa}\big] \frac{1}{(1-2\kappa\cos\theta+\kappa^2)^{\frac{1}{2}}}\,. \tag{44b}$$

The product of the far-field functions follows from the definition (38) as

$$f^{qp}(\theta) f^{pq}(\theta) = \sum_{n=0}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_{\ell=0}^{+\infty} (2n+1)(2\nu+1) T_n^{qp} T_\nu^{pq} P_\ell(\cos\theta) G(0,\nu;0,n;\ell). \tag{45}$$

Combing the above results and performing the differentiation in (44b) gives

$$S(\kappa) = \frac{1}{2}(1-\kappa^2) \int_0^\pi f^{qp}(\theta) f^{pq}(\theta) \big(1 - 2\kappa\cos\theta + \kappa^2\big)^{-\frac{3}{2}} \sin\theta \,\mathrm{d}\,\theta. \tag{46}$$

The series $S(\kappa)$ is therefore convergent since it can be expressed as the integration of a continuous function over the compact interval $[0,\pi]$. Although we did not manage to express the remaining series in Eq. (36) in terms of the form functions when $|\kappa|>1$, the series are always convergent because there is only a limited number of significant values of the T_n^{qp} coefficients, see the discussion on this point in Ref.[5]. It is also easy to show that the Gaunt coefficients decrease much more quickly that the function $|\kappa|^\ell$ increases with $\ell\to+\infty$ even for $|\kappa|>1$.

If p indicates the faster wave propagating in the medium, then it is always true that $|\kappa|=|k_p/k_q|<1$ whatever the value of $q\neq p$. This is the case for acoustic waves in viscous fluids and compressional waves in viscoelastic solids. So, in such cases, if ξ_p is the wavenumber of the acoustic or compressional wave, it follows in a straightforward manner from Eqs. (36) to (46) that

$$\xi_p^2 = k_p^2 + \delta_1 n_0 + \delta_2 n_0^2 + \delta_2^{(c)} n_0^2 \tag{47}$$

with

$$\begin{aligned}\delta_2^{(c)} &= \sum_{q\neq p} \frac{16\pi^2}{k_p k_q (k_q^2-k_p^2)}\, S(\frac{k_p}{k_q}) \\ &= \frac{8\pi^2}{k_p}\sum_{q\neq p}\int_0^\pi \frac{f^{qp}(\theta) f^{pq}(\theta)\,\sin\theta\,\mathrm{d}\,\theta}{\left(k_p^2+k_q^2-2k_pk_q\cos\theta\right)^{\frac{3}{2}}}.\end{aligned} \tag{48}$$

The additional term $\delta_2^{(c)}$ indicates the coupling between the compressional (p), shear and thermal waves ($q\neq p$). The formula given by Eqs. (47) combined with (48) generalizes the identity for acoustic waves derived by Lloyd and Berry[1] for which $\delta_2^{(c)}=0$. It is of interest to compare the structure of $\delta_2^{(c)}$ with that of the corresponding term for the effective quasi-longitudinal wave in the presence of cylinders in an elastic host medium[5] (Theorem 1). The coefficient for the latter case is obtained from (48) by removing $\sin\theta$ in the numerator and replacing the power $\frac{3}{2}$ in the denominator with 1, that is, by making the obvious changes one would expect for 2D as compared with 3D.

Appendix A: Azimuthal waves at second order

The aim of this appendix is to show that the coherent fields do not depend on the azimuthal direction φ up to second order in concentration. We refer here to equations in the paper of Linton and Martin[2]. The fact that it only considers compressional or acoustic waves ($P=1$) is not important for the present demonstration, which is the same for one or for several waves ($P=3$). We begin with the modal equation for oblique incidence, (*cf.*[2] (Eq. 4.20))

$$F_n^m + \frac{in_0(4\pi)^2(-1)^m}{k(k^2-K^2)}\sum_{\nu=0}^{+\infty}\sum_{\mu=-\nu}^{+\nu}\sum_{q=0}^{+\infty} Z_\nu F_\nu^\mu Y_q^{\mu-m}(\widehat{\mathbf{K}})\mathbf{N}_q(Kb)\mathbf{G}(n,m;\nu,-\mu;q)=0, \tag{A1}$$

with the spherical harmonics and the Gaunt coefficients defined by, respectively,

$$Y_n^m(\widehat{\mathbf{r}}) = Y_n^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{im\varphi}, \tag{A2a}$$

$$Y_n^m(\widehat{\mathbf{r}})Y_\nu^\mu(\widehat{\mathbf{r}}) = \sum_{q=0}^{\infty} Y_q^{m+\mu}(\widehat{\mathbf{r}})\mathbf{G}(n,m;\nu,\mu;q). \tag{A2b}$$

Consequently, it is necessary to prove that the unknown coefficients F_n^m are zero except for $m=0$, at least up to order 2 in concentration. We use the following result for the expansion of F_n^m up to and including the second order in concentration (*cf.*[2] (Eq. 4.20) and its preceding equation)

$$F_n^m = \overline{Y}_n^m(\widehat{\mathbf{K}})\tilde{F} + n_0\{\overline{Y}_n^m(\widehat{\mathbf{K}})V + \frac{(4\pi)^2 b}{2k^2}(-1)^m\tilde{F}\sum_{\nu=0}^{+\infty}\sum_{\mu=-\nu}^{+\nu} Z_\nu \overline{Y}_\nu^\mu(\widehat{\mathbf{K}})X_{n\nu}^{m\mu}\} \tag{A3}$$

with[2] (Eq. 4.27)

$$X_{n\nu}^{m\mu} = \sum_{q=0}^{+\infty} Y_q^{\mu-m}(\widehat{\mathbf{K}})\mathbf{G}(n,m;\nu,-\mu;q)\,d_q(kb), \tag{A4}$$

where $\tilde{F}$ is a constant and where V^2 (Eq. 4.31) and $d_q(kb)^2$ (Eq. 4.23) are functions which do not depend on the indices n and m. Since the coherent wavenumbers do not depend upon the angle of incidence[2], we may consider the case of normal incidence for which $\widehat{\mathbf{K}} = (0,0,1)^2$ (Eq. 3.15), so that

$$Y_q^{\mu-m}(\widehat{\mathbf{K}}) = \sqrt{\frac{2q+1}{4\pi}}\delta_{\mu m}. \tag{A5}$$

Equation (A3) therefore reduces to

$$\begin{aligned} F_n^m =& \sqrt{\frac{2n+1}{4\pi}}\delta_{m0}\left(\tilde{F} + n_0 V\right) + n_0 \tilde{F}\frac{4\pi b}{2k^2}(-1)^m \\ &\times \sum_{\nu=0}^{+\infty}\sum_{\mu=-\nu}^{+\nu}\sum_{q=0}^{+\infty} Z_\nu \sqrt{(2\nu+1)(2q+1)}\delta_{\mu 0}\,\delta_{\mu m}\,\mathbf{G}(n,0;\nu,0;q)d_q(kb), \end{aligned} \tag{A6}$$

which proves that $F_n^m = 0$ if $m \neq 0$ up to second order in concentration.

Consequently, at normal incidence and up to order 2 in concentration, Eq. (A1) becomes

$$F_n^0 + \frac{in_0(4\pi)^2}{k(k^2-K^2)}\sum_{\nu=0}^{+\infty}\sum_{q=0}^{+\infty} Z_\nu F_\nu^0 \sqrt{\frac{2q+1}{4\pi}}\mathbf{N}_q(Kb)\mathbf{G}(n,0;\nu,0;q) = 0. \tag{A7}$$

Our results correspond exactly to those of Ref.[2]. Indeed, we can easily show that Eq. (A7) and Eq. (18) with $P=1$ are exactly the same if we make the following identifications: $k = k_1$, $K = \xi_1 = \xi$, $Z_\nu = -T_\nu^{11}$, $\mathbf{N}_q(Kb) = ik_1 b N_n^{(1)}(\xi)$ and

$$F_n^0 = (-1)^n\sqrt{\frac{2n+1}{4\pi}}A_n^{(1)}, \quad \mathbf{G}(n,0;\nu,0;q) = G(0,\nu;0,n;q)\sqrt{\frac{(2n+1)(2\nu+1)}{4\pi(2q+1)}}. \tag{A8}$$

Appendix B: Some asymptotic expansions

With no loss of generality we consider the first root y_1. It follows from Eqs. (16), (29), and (30) that the asymptotic expansion of the 11 element of the matrix in Eq. (27) is

$$y_1 - \varepsilon M_{11} = \left[y_1^{(1)} - M_{11}^{(0)}(k_p)\right]\varepsilon + \left[y_1^{(2)} - M_{11}^{(1)}(k_p)\right]\varepsilon^2 + \dots\,. \tag{B1}$$

Hence, to leading order Eq. (27) becomes

$$\begin{vmatrix} y_1^{(1)} - M_{11}^{(0)}(k_p) & 0 & 0 \\ -M_{12}^{(0)}(k_p) & k_1^2 - k_2^2 & 0 \\ -M_{13}^{(0)}(k_p) & 0 & k_1^2 - k_3^2 \end{vmatrix} = 0, \tag{B2}$$

which implies the identity (30a). Inserting the latter into Eq. (B1) and Eq. (27) then gives, at the leading order,

$$\begin{vmatrix} y_1^{(2)} - M_{11}^{(1)}(k_p) & -M_{21}^{(0)}(k_p) & -M_{31}^{(0)}(k_p) \\ -M_{12}^{(0)}(k_p) & k_1^2 - k_2^2 & 0 \\ -M_{13}^{(0)}(k_p) & 0 & k_1^2 - k_3^2 \end{vmatrix} = 0, \tag{B3}$$

from which Eq. (30b) follows.

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